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Magnetic moments of branes and giant gravitons

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ABSTRACT: We study the magnetic analogue of Myers' Dielectric Effect and, in some cases, relate it to the blowing up of particles into branes, first investigated by Greevy, Susskind and Toumbas. We show that D0-branes or gravitons in M-theory, moving in a magnetic four-form field strength background expand into a non-commutative two sphere. Both examples of constant magnetic field and non-constant fields in curved backgrounds generated by branes are considered. We find, in all cases, another solution, consisting of a two-brane wrapping a classical two-sphere, which has all the quantum numbers of the D0-branes. Motivated by this, we investigate the blowing up of gravitons into branes in backgrounds different from $AdS_m \times S^n$. We find the phenomenon is quite general. In many cases with less or even no supersymmetry we find a brane configuration which has the same quantum numbers and the same energy as a massless particle in supergravity.

KEYWORDS: M-Theory, D-branes, Black Holes in String Theory, AdS/CFT Correspondance.

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Contents

1. Introduction and summary

We are getting increasing familiar with the idea that in string theory particles grow in transverse size with increasing energy [1]. This idea is supported by the stringy uncertainty principle [2]. and the IR/UV connection. Another important and related development is that of non-commutativity [3, 4, 5] — the idea that space-time coordinates do not commute with each other.

An interesting example of the growth in size with energy was found recently in [6]. These authors studied a graviton[1] in $AdS_m \times S^{p+2}$ which rotated on the S^{p+2} and carried angular momentum. The graviton is a BPS state and has an energy equal to its angular momentum. Somewhat surprisingly, [6] showed that

[1]More correctly we mean an appropriate supergravity mode. Throughout this paper we will loosely refer to such modes as gravitons.

the same BPS relation is satisfied by an expanded brane configuration. For large angular momenta, [6] argued, the graviton blows up into the expanded brane whose size increases with increasing angular momentum for $p \geq 2$. Since the size of the expanded brane is bounded by the radius of the S^{p+2} there is a maximum bound on the angular momentum; this agrees with the stringy exclusion principle [7].

The phenomenon described above is quite similar to Myers' Dielectric effect [8]. It was found in [8] that due to the non-abelian nature of their world volume theory N D0-branes placed in an electric RR four-form field strength expand into a non-commutative two-sphere. There is another solution in the theory consisting of a D2-brane which wraps the corresponding classical two-sphere. The D2-brane carries N units of U(1) magnetic field in its world volume and has exactly the same quantum numbers as the D0-brane configuration.

This paper explores the relation between [6] and [8] and extends the analysis of [6] to more general settings.

We begin by demonstrating the magnetic analogue of the Dielectric Effect (we will refer to this as the Magnetic Moment effect below). A simple setting is provided by a constant four form magnetic field, F^4_{1234} in type-II theory. One finds that D0-branes, when moving in the magnetic field, blow up into a non-commutative two-sphere. The size of this sphere increases with increasing momentum. We also find a minimal energy D2-brane solution which has the same momentum and wraps the corresponding classical two-sphere. Both the puffed D0-branes and D2-brane carry a magnetic dipole moment with respect to F^4. Variants of the Dielectric effect were studied in [9] leading to fuzzy $S^2 \times S^2$, CP^2 and SU(3)/(U(1) $\times$ U(1)). Their magnetic analogues are also discussed. The resulting configurations can carry dipole or quadropole magnetic moments.

D0-branes are gravitons in M-theory. This leads us to consider a situation where F^4_{123M} is non-zero and the graviton moves along the M direction. In the type-IIA limit this reduces to static D0-branes in a constant NS H_{123} field. Once again, following, [8], we show that the D0-branes "puff up" into a non-commutative two-sphere. The size of this sphere grows with increasing momentum in the M direction. There is an alternative description of this configuration in terms of a two-brane wrapping a two-sphere.

Once the Magnetic Moment Effect is understood for constant magnetic field one can study it in more complicated examples. Towards the end of the paper we study D0-branes moving in the background of a D4-brane. In this case the background geometry is curved and the four-form field strength threads a four-sphere (of varying radius). We show that once again rotating D0-branes expand into a two-sphere. Moreover the resulting configuration has exactly the same energy as if the D0-branes were executing only center of mass motion with no relative displacement. We also find a D2-brane solution with U(1) flux on its worldvolume, which has exactly the same quantum numbers and in fact (in the appropriate limit) the same energy.

JHEP10(2000)037

The understanding of the Magnetic Moment Effect prompts us to extend the analysis of [6] to more general settings. In a sense the results of [6] are surprising since one finds that expanded brane configurations — which are normally thought to be heavy objects — can have the same energy as massless particles. This happens because the coupling to the magnetic field threading the $(p+2)$ sphere precisely cancel the effects of the brane tension. At first sight this appears to be a very special feature of a particular kind of motion in $AdS_m \times S^{p+2}$ space-times. It was shown in [10] that the energy for arbitrary brane motions obeys a BPS *bound*[2] and the special motion considered in [6] (i.e. motion without change of size and without oscillations on the brane) saturates the bound. Furthermore in a supersymmetric theory these motions have been shown to preserve half of the supersymmetries of the background [11] so that the BPS bound follows from supersymmetry. The derivation of the BPS bound in [10] follows from a delicate cancellation which depends on detailed form of the background and one might wonder whether the result has any level of universality.

We find that gravitons can turn into expanded branes in other spacetimes as well. Contrary to the expectations mentioned above it turns out that in several cases, including spacetimes with no supersymmetry at all, the brane solution has the same energy as the graviton. More specifically, we study gravitons in various extremal and non-extremal $(6-p)$-brane backgrounds. These backgrounds preserve a SO$(p+2)$ subgroup of the R-symmetry group and the gravitons carry SO$(p+2)$ charge. In the near horizon limit, for both the extremal and *non-extremal* cases, we find that a p-brane configuration, which is the solution of least energy for a given angular momentum, has exactly the same energy and motion as a graviton. Moving away from the near horizon limit the energies do not agree anymore. The extremal background geometry, for $p \neq 3$ has half the supersymmetries as the AdS cases studied in [6]. As best as we can tell, the Dp-brane configuration moving in this background does not preserve any of them.[3] The non-extremal geometry clearly breaks all supersymmetries. Similar results also hold for the non-extremal M2- and M5-branes.

Our considerations also apply to the extremal and non-extremal geometries of five and four-dimensional black holes in string theory obtained by compactification on $\mathbb{T}^5$ and $\mathbb{T}^6$, respectively. Once again one finds brane configurations with the same energy as gravitons. We should add though that the interpretation of the brane configurations in these cases as gravitons is not so clear.

[2]We use the phrase BPS bound in the original sense of term, viz. the fact that the energy is bounded from below by a conserved charge

[3]In these examples, there is generically no BPS bound, except for $p = 3$. We work in Poincare coordinates. In such coordinates a graviton in the near horizon region with some angular momentum is not static but would fall into the "center". Only for $p = 3$ one can go to a different coordinate system — global coordinates in AdS_5.

It is worth emphasising that expanded p-brane configurations with the same energy as massless particles can be found only if an important condition is satisfied. The p-brane in the course of its motion sweeps out a $S^p \times S^1$ surface. The metric seen by the p-brane is related to the string metric by a conformal rescaling. For Dp-branes the condition says that the volume of this surface in the p-brane metric must equal (in appropriate string units) the number of $(6-p)$-branes. In the M-theory cases a similar condition must hold without any conformal rescaling. Unfortunately, the significance of this condition is unclear to us at the moment. It is worth pointing out though that in all cases the p-brane which acts like a giant graviton is the magnetic dual of the brane which produce the geometry.

One can be justified in claiming that gravitons turn into expanded branes only if the two descriptions, of graviton and expanded brane are not simultaneously valid. In section 5 we show that this is indeed true. The graviton description is controlled, in the supergravity approximation, for small angular momenta, when higher derivative corrections to the supergravity lagrangian can be neglected. In contrast the expanded brane description is valid for large angular momenta, when the size of the brane is large, and its curvature is small, so that the Born Infeld action which neglects higher derivative terms can be used. We establish this both for the AdS case studied in [6] and for the general p-brane backgrounds considered in this paper.

Our study is incomplete in some important respects.

The Magnetic Moment Effect discussed here provides a quantitative understanding of expanded branes in only one case: D0-branes rotating in the D4-brane background, which expand into a D2-brane with magnetic flux, as was mentioned above. Such an understanding for expanded branes without a world volume magentic flux is missing though. In particular we do not have a good understanding of the blowing up of gravitons in $AdS_m \times S^{p+2}$ or Dp-brane backgrounds. In the $AdS_m \times S^{p+2}$ case, for $(m,n) = (4,7)$, and, $(7,4)$ though, the Magnetic effect does provide at least a qualitative understanding. In these cases if we are cavalier and regard one of directions of the S^n to be the M-theory direction, gravitons moving in this direction are D0-branes. From the IIA viewpoint these D0-branes are in a background of a NS 3-form field strength for $(m,n) = (7,4)$ and a RR 6-form field strength for $(m,n) = (4,7)$. In the former case one would expect the D0-branes to blow up into fuzzy two-spheres as shown in this paper. In the latter case, the couplings indicate that there could be solutions with non-commutative four-spheres, though this is not quite understood yet.

Another important question which remains is to understand the blowing up of gravitons or D0-branes in the dual holographic gauge theory. For example, the D0-branes mentioned above in the D4-brane background are Yang Mills instantons in the boundary theory. One can ask: what is the holographic description of their blowing up into D2-branes?[4] More generally one can ask the same question about gravitons and other massless states in the gravity theory. The fact that the giant graviton phenomenon is more general may be important in understanding the structure of spacetime at short distance scales. It has been also argued that the stringy exclusion principle and some of its manifestations [12] means that the dual supergravity should live on a non-commutative space-time, e.g. quantum deformations of $AdS \times S$ [13], for a related effect in matrix theory see [14]. It has been suspected that the dynamics of giant gravitons, in particular the upper bound on the angular momentum for special class of states, point to such a noncommutativity — a connection which has been explored in [10] and [15]. A holographic description would perhaps relate this spacetime Non-commutativity to Non-commutativity in the boundary theory.

Finally, the process by which D0-branes or gravitons turn into the corresponding expanded brane, seems related to the decay of brane-antibrane pairs into lower dimensional branes [16]. Investigating the connection in more detail would be worthwhile.

2. The magnetic moment effect

2.1 The electric Myers effect

We begin by briefly recalling the electric Myers effect [8] (the fuzzy sphere in matrix theory was considered in [17]). Consider D0-branes in a transverse electric four form field strength background:

$$F^{(4)}_{0ijk} = \begin{cases} -F\epsilon_{ijk}\,, & \text{for } i,j,k \in \{1,2,3\}; \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.1}$$

where F is a constant. For a static configuration, the resulting D0-brane Energy is given by

$$E = T_0 N - \frac{T_0}{4\lambda^2}\sum_{ab} \mathrm{Tr}([X^a, X^b]^2) - i\frac{T_0}{3\lambda}\,\mathrm{Tr}(X^i X^j X^k) F^{(4)}_{0ijk}\,. \tag{2.2}$$

The last term in (2.2) was discussed in [18] and [8]. In our conventions

$$T_p = \frac{2\pi}{g_s(2\pi l_s)^{p+1}}\,, \qquad \lambda = 2\pi l_s^2\,. \tag{2.3}$$

For static configurations, one can show that the energy is minimized by setting,

$$X^i = \frac{\lambda F}{2} J^i\,, \qquad i = \{1,2,3\}\,, \tag{2.4}$$

[4] The analogous question for branes expanding in AdS space was studied in [11]. We should mention that in this paper we will use Poincare coordinates as opposed to global coordinates. The boundary theory in this case lives in flat space and does not have an $R\phi^2$ term coupling. Correspondingly there are no finite energy states in the bulk with branes extended in the AdS directions.

where J^i denote N-dimensional representation of SU(2), with the remaining coordinates being proportional to the identity matrix. The solution (2.4) is a fuzzy two-sphere. Choosing J^i to be in the N-dimensional irreducible representation of SU(2) gives a radius and energy for the solution (2.4):

$$\begin{aligned} R &= \frac{\lambda}{2} F \sqrt{\frac{N^2-1}{4}} \simeq \frac{\lambda}{4} F N \,, \\ E &= T_0 N - \frac{T_0}{96} \lambda^2 F^4 N \frac{N^2-1}{4} \simeq T_0 N - \frac{T_0}{384} \lambda^2 F^4 N^3 \,, \end{aligned} \tag{2.5}$$

where the two approximate equalities relate to the $N \gg 1$ limit. We also note that the resulting configuration carries a dipole moment with respect to the four form field strength.

Another configuration with the N units of D0-brane charge and the same dipole moment with respect to $F^{(4)}$ can be constructed in terms of one D2-brane wrapping a sphere in the X^1, X^2, X^3 directions. The D2-brane carries N units of U(1) world volume magnetic flux. The energy for such a static brane, which follows from the Dirac Born Infeld action and the Cherns Simons terms, (for a review see [19]) is given by

$$E = 4\pi T_2 \sqrt{r^4 + \frac{N^2}{4}\lambda^2} - \frac{4\pi}{3} T_2 F r^3 \,, \tag{2.6}$$

where r denotes the radius of the two-sphere and we have substituted for $F^{(4)}$ from (2.1). Notice, that the energy does not have a global minimum and goes to $-\infty$ as $r \to \infty$. This indicates an instability for the two-brane to grow very big.

There can, however, be a local minimum for a suitable range of parameters. When

$$r^2 \ll N\lambda \,, \tag{2.7}$$

(2.6) can be expanded as:

$$E \simeq 2\pi T_2 \lambda N + \frac{4\pi T_2}{\lambda} \frac{r^4}{N} - \frac{4\pi}{3} T_2 r^3 F \,. \tag{2.8}$$

This gives a minimum at a radius

$$R = \frac{\lambda}{4} F N \,, \tag{2.9}$$

and an energy equal to (2.5). Consistency with (2.7) imposes the condition:

$$N\lambda F^2 \ll 1 \,. \tag{2.10}$$

More generally the full energy (2.6) needs to be minimised. One can show that a local minimum only exists if

$$F^2 < \frac{4}{N\lambda} \,. \tag{2.11}$$

A few comments are worth making at this point.

The expression for the energy (2.2) is an approximation; in general there are additional terms involving higher powers of the transverse coordinates. This approximation is justified only if the radius of the two-sphere is small compared to the string scale (the masses of the "W" bosons are then small in string units). In contrast for the D2-brane, the coordinates along the two-sphere lie along the world volume. The DBI action which gives rise to the energy, (2.6), is a good approximation when the size of the two-sphere is big compared to the string scale. In this limit higher derivative terms — that is "acceleration terms" — can be neglected. Thus we see that, in general, the two descriptions, in terms of the puffed up D0-branes and the wrapped D2-brane, are valid in different regions of parameter space. Not surprisingly, the energy and radius of the fuzzy sphere derived from both descriptions do not agree.

Agreement is obtained in the limit (2.7) though. In fact in this limit, not only does the radius and the energy of the D2-brane agree with that of (2.5), but each terms of the expansion (2.8), agrees with (2.2). A little thought, along the lines of [4], shows that this agreement is to be expected. The important point is that the D0-brane desription is valid when the two-sphere has a radius, measured in the *closed string metric*, which is small compared to the string scale, while the D2-brane action is valid when the radius, in the *open string metric*, is big compared to the string scale. When (2.7) is true, an argument similar to that in [4], shows that both requirements are met simultaeneously (for large N).

Finally, we have neglected the curvature of the spacetime due to the $F^{(4)}$ field strength. Strictly speaking, this back reaction[5] needs to be incorporated.[6] Our neglect can be justified if the theory under consideration is a boundary hologram, or perhaps, if there are other sources, besides $F^{(4)}$, cleverly turned on to keep the metric flat and dilaton constant. Later on, in the context of the magnetic effect, we will consider examples where the back reaction is included.

2.2 The magnetic moment effect

Consider a four form background of the form:

$$F^{(4)}_{ijkl} = \begin{cases} -F\epsilon_{ijkl}, & \text{for } i,j,k,l \in \{1,2,3,4\}; \\ 0 & \text{otherwise}. \end{cases} \tag{2.12}$$

The background preserves a SO(4) × SO(5) symmetry. The resulting lagrangian is now

$$L = -T_0 N + \frac{T_0}{2}\,\mathrm{Tr}(\dot{X}^i)^2 + \frac{T_0}{4\lambda^2}\sum_{ab}\mathrm{Tr}([X^a,X^b]^2) + i\frac{T_0}{3\lambda}F^4_{ijkl}\,\mathrm{Tr}[X^iX^jX^k(\dot{X}^l)]\,. \tag{2.13}$$

[5] An earlier draft of the paper contained an erroneous statement in this context. We thank Andrew Frey for bringing this to our attention.

[6] Note, in our conventions, (2.2) the action goes like $S = -\int \frac{1}{g_s^2}(F^4)^2 + \cdots$.

The derivative above is with respect to time. The last term is linear in the velocity as is usual in a coupling to the magnetic field, it was also considered in [18].

One can show that the equations of motion which follow from (2.13) and (2.12) can be solved by

$$X^4 = vX^0 \mathbb{1}\,, \qquad X^i = \frac{\lambda}{2} F v J^i\,, \qquad i = \{1,2,3\}\,, \tag{2.14}$$

with all other X^a's being constant and proportional to the identity matrix. Other solutions to (2.13) and (2.12) can be obtained by performing a SO(4) rotation on the four coordinates.

The D0-branes have thus expanded into a non-commutative two-sphere in the directions transverse to the velocity. The radius of this two-sphere depends linearly on the velocity and the four form field strength. If we choose the J^i matrices in (2.14) to be in the N-dimensional irreducible representation the radius is

$$R = \frac{\lambda}{2} F v \sqrt{\frac{(N^2-1)}{4}} \sim \frac{\lambda}{4} F v N\,, \tag{2.15}$$

where the approximate equality is valid for large N. The energy for this case (when $N \gg 1$) is

$$E = NT_0 + \frac{T_0}{2} N v^2 - \frac{T_0}{128} \lambda^2 F^4 v^4 N^3\,. \tag{2.16}$$

It is also interesting to express the radius R in terms of the momentum along the X^4 direction, P_4. This follows by noting that

$$P_4 = NT_0 v + \frac{2}{3} \frac{F R^3 T_0}{\lambda}\,, \tag{2.17}$$

eq. (2.15) then implies that

$$P_4 = \left(\frac{4T_0}{\lambda F}\right) R + \left(\frac{2T_0 F}{3\lambda}\right) R^3\,. \tag{2.18}$$

Note that N has dropped out from (2.18).

For reasons mentioned in the the electric case, the action (2.13) is a good approximation when the radius (2.15) (measured in the closed string metric) is small in string units. We should also mention that choosing a reducible representation in (2.14) gives rise to more than one fuzzy sphere, in general of different radii.

Next, consider a D2-brane wrapping a sphere of radius r in the $1,2,3$ directions and moving in the X_4 direction. Its action is given by

$$S = \int dt \left[-4\pi T_2 \sqrt{1 - (\dot{r})^2 - (\dot{X}^4)^2} \sqrt{r^4 + \frac{N^2\lambda^2}{4}} + \frac{4\pi}{3} T_2 r^3 F \dot{X}^4 \right]. \tag{2.19}$$

The terms above within the square root arise from the Born Infeld action while the last term comes from the Cherns Simon action which has the form

$$S_{\mathrm{CS}} = T_p \int C^{p+1}\,, \tag{2.20}$$

for a p-brane. Assuming the motion is non-relativistic and that $r^2 \ll N\lambda$, one has,

$$\begin{aligned} S \simeq \int dt\, \Big[& -2\pi T_2 \lambda N + 2\pi T_2 N\lambda(\dot{r})^2 + 2\pi T_2 N\lambda(\dot{X}^4)^2 - \frac{4\pi T_2}{N\lambda} r^4 + \\ & + \frac{4\pi}{3} T_2 r^3 \dot{X}^4 F \Big]\,. \end{aligned} \tag{2.21}$$

Putting in the ansatz for a non-commutative two-sphere of radius r in (2.13), one finds that (2.13) and (2.21) agree term by term with $\dot{X}^4$ being identified with v. Thus setting, $\dot{r}$ equal to zero and minimizing the action with respect to r gives a radius and an energy from (2.21) which agrees with (2.15) and (2.16).

However, in the more general case when the motion is relativistic or $r^2 \geq N\lambda$, one needs to work with the full action, (2.19). It is useful then to discuss the dynamics in terms of the hamiltonian. We have:

$$\begin{aligned} P_r &= \frac{\partial L}{\partial \dot{r}} = 4\pi T_2 \sqrt{r^4 + \frac{N^2\lambda^2}{4}} \frac{\dot{r}}{\sqrt{1 - (\dot{r})^2 - (\dot{X}^4)^2}}\,, \\ P_4 &= \frac{\partial L}{\partial \dot{X}^4} = 4\pi T_2 \sqrt{r^4 + \frac{N^2\lambda^2}{4}} \frac{\dot{X}^4}{\sqrt{1 - (\dot{r})^2 - (\dot{X}^4)^2}} + \frac{4\pi}{3} T_2 r^3 F\,. \end{aligned} \tag{2.22}$$

The hamiltonian is

$$H = \left[(4\pi T_2)^2 \left(r^4 + \frac{N^2\lambda^2}{4} \right) + P_r^2 + \left(P_4 - \frac{4\pi}{3} T_2 r^3 F \right)^2 \right]^{1/2} \tag{2.23}$$

P_4 is a constant of motion. Restricting to motion with constant radial size, we set $P_r = 0$ and minimize H with respect to r. It is easy to check that apart from the trivial solution $r = 0$ there is always a single real solution of this equation with non-zero r. (2.16) is the non-relativistic limit of (2.23), therefore, the energy that follows from (2.23) is in general different from (2.16). Interestingly though, the radius that follows from (2.23) agrees with (2.18).

Let us end with a few comments. First, in the magnetic case there is a one parameter family of solutions depending on v in (2.16) or P_4 in (2.23). The transverse size of the sphere depends on this parameter. In fact, from (2.23) it is clear that the equilibrium radius depends on P_4 only and not on N, consistent with (2.18). Second, one could have guessed the form of the solution for the magnetic case, from electric one by performing a boost. But strictly speaking one cannot go from the

JHEP10(2000)037

purely electric to purely magnetic case by a boost. Third, as in the electric case we have neglected the backreaction on the metric due the RR field strength. We will comment, briefly, on this issue in the next section. Fourth, the resulting fuzzy two-sphere carries an electric dipole moment which couples to the electric four-form field strength. It also has a magnetic dipole moment which couples to the magnetic four-form field strength. The induced magnetic dipole moment results in lowering the energy of the configuration (for fixed momentum P_4), as seen in the last term in (2.16) or (2.23). In this sense the system behaves like a paramagnet. The decrease in energy — which goes like N^3 in (2.16) — can be significant for large N

Finally, generlisations of the Dielectric effect which yield fuzzy $S^2\times S^2$, CP^2 and $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$ were studied in [9]. The corresponding magnetic generalisation are straightforward. Replace (2.12) by

$$F^{(4)}_{ijk9} = -F f_{ijk}\,, \tag{2.24}$$

with all other components being zero. Taking f_{ijk} to be the structure functions for $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and moving the D0-branes in the X^9 direction, one finds that the D0-branes have puffed up into a fuzzy $S^2\times S^2$, with a radius which is again linearly dependent on the velocity. Similarly, choosing f_{ijk} to be the structure functions for SU(3), one finds, depending on the choice of irreducible representation, fuzzy CP^2 or $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$. In all these cases there is also a description in terms of a higher dimension expanded brane; a D4-brane for $S^2\times S^2$ and CP^2 and a six-brane for $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$. These expanded branes have a world volume U(1) gauge field turned on and carry the same quantum numbers as the puffed up D0-branes [9]. They also have induced electric and magnetic multipole moments. The $S^2\times S^2$ configuration has electric and magnetic quadropole moments, while CP^2 and $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$ have electric and magnetic dipole moments.

3. Giant gravitons in M-theory

D0-branes in M-theory are gravitons moving along the M direction. The example we considered above for the magnetic case can be interpreted in M-theory as a situation where the gravitons move both along the M direction and along X^4. In fact the action (2.19) and hence the hamiltonian (2.23) are precisely those of a M2-brane with a momentum $P_M = N/gl_s$ in the M direction (where g is the string coupling).

It is natural to assume that the simpler situation where the graviton moves along say only the M direction with $F^{(4)}_{123M}$ turned on would also result in the graviton expanding into a transverse sphere. The M2-brane in turn should be transverse to and moving along the M direction and should be expanded along the classical two-sphere.

We will see next that this is indeed true. To keep the description for the graviton and two-brane under control we analyze this in the type-IIA limit first. In type-IIA one is looking for a solution consisting of N static D0-branes subject to an external H_{123} field. In fact this situation was considered in [8]. The energy for this static configuration is

$$E = NT_0 - \frac{T_0}{4\lambda^2}\sum_{ab}\mathrm{Tr}([X^a, X^b]^2) - i\frac{T_0}{3\lambda}H_{ijk}\,\mathrm{Tr}(X^i X^j X^k)\,. \tag{3.1}$$

Setting,

$$H_{123} = -F\,, \tag{3.2}$$

(with all other components, not related by symmetries, equal to zero) in (3.1) one sees that this is in fact identical to (2.2) above. Thus the resulting solution which minimises the energy is a non-commutative two sphere and the D0-branes (equivalently the M-theory graviton) are indeed puffed up in the presence of the external field. The radius and energy of the configuration is given by (2.5).

To analyze this situation from the two-brane point of view we use the description in terms of the D2-brane action. As was mentioned above the M2-brane is expected to be transverse to the M direction. In this case, the D2- and M2-brane world volume theories are related by a duality transformation which turns the scalar field corresponding to the M direction in the M2-brane world volume theory, into the D2-brane gauge field. Thus we expect the D2-brane theory to have N units of magnetic flux. The energy for a static D2-brane with H_{123} turned on is

$$E = 4\pi T_2\sqrt{r^4 + \left(\frac{N\lambda}{2} - \frac{1}{3}Fr^3\right)^2}\,. \tag{3.3}$$

This is different from (2.6). The radius of this brane configuration can be obtained by minimising (3.3) with respect to r. Note that in (3.3) unlike (2.6), the energy grows as r^3 for large r so a minimum exists for all values of F. In general the energy and radius we obtain will not agree with (2.5). However, in the limit when $r^2 \ll N\lambda$ the square root can be expanded in (3.3) and once again yields the three terms of (2.8). Thus the energy and radius agree with (2.5).

There is one important issue to be noted in our discussion above. We have neglected the backreaction of H_{123} on the metric. We leave a full discussion after including backreaction effects for the future and content ourselves here with some estimates. The metric perturbation induced by (3.2) over a region of size R is

$$\delta h_{\mu\nu} \sim R^2F^2\,. \tag{3.4}$$

The same estimate also applies for the dilaton. It is useful to start in the limit when the back-reaction is small, i.e.

$$R^2F^2 \ll 1\,. \tag{3.5}$$

In this limit one can make a self-consistent estimate and argue that

$$R \sim \lambda F N\,, \tag{3.6}$$

this is the same order of magnitude as (2.9). The argument goes as follows. Let us assume that (3.6) is true. Then from (3.1) and (3.6) one can argue that the the leading order contribution to the energy is

$$E_0 = NT_0 \,. \tag{3.7}$$

The first corrections to this is of order

$$E_1 \sim T_0 \lambda^2 F^4 N^3 \,. \tag{3.8}$$

The leading contributions from the second and third terms in (3.1) are of this order and the back-reaction can be neglected in obtaining them. However, the back-reaction is important in obtaining the subleading contribution from the first term in (3.1). This term depends on the dilaton through the D0-brane tension. Taking into account the back-reaction in the dilaton of order (3.4) yields its contribution, which is also of order (3.8). The resulting three terms of order (3.8) must be minimised to yield a radius. One expects an answer of order (3.6), since all three terms are then comparable.

From the expanded D2-brane point of view, note that (3.6) and (3.5) imply that $R^2 \ll N\lambda$. So the limit when the back reaction is small, is precisely the limit discussed above when the square root in (3.3) can be expanded resulting in three terms which correspond to (3.1). When the backreaction is neglected, we showed above that these three terms agree quantitatively with those in (3.1). When the backreaction is included one can show that a similar agreement persists in the limit (3.6) and (3.5).

4. Giant gravitons in brane backgrounds

We have seen above that M-theory gravitons in appropriate background fields can turn into expanded branes. The phenonemon was investigated in AdS space in [6], as was mentioned in the introduction. It was found that the hamiltonian of a p-brane moving on the $p+2$ sphere of a $AdS_m \times S^{p+2}$ space-time (and not performing any other kind of motion or oscillation) is *exactly* the same as that of a massless particle with the same quantum numbers. It is rather remarkable that a "heavy" object like a brane can have an energy with a gapless spectrum. The reason behind this is a delicate cancellation of the effect of the brane tension with the energy due to coupling to the background F_{p+2} form gauge field leading to a BPS like condition [10]. Furthermore the configurations which saturate this BPS bound also preserve half of the supersymmetries of the background [11]. This mechanism seems to depend on the details of the background geometry and makes one wonder whether it is a phenomenon restricted to $AdS_m \times S^{p+2}$ spacetimes.

In this section we will show that blowing up of gravitons into expanded branes with the same energy is much more general and occurs in a wide variety of spacetimes.

JHEP10(2000)037

The backgrounds we consider are those of both extremal and non-extremal branes. Significantly this includes backgrounds with no supersymmetry.

4.1 Dp-branes in background of N D$(6-p)$-branes

To keep the discussion general we consider a $(6-p)$-brane geometry with a metric of the form:

$$ds^2 = -g_{tt}dt^2 + \sum_{i=1}^{6-p} g_{ii}(dX^i)^2 + g_{rr}dr^2 + f(r)r^2 d\Omega_{p+2}^2 \,. \tag{4.1}$$

This metric has an SO$(p+2)$ rotational symmetry and we will in particular be interested in states which carry SO$(p+2)$ angular momentum. In the discussion below it will be useful to choose the following coordinates on a unit $p+2$ sphere:[7]

$$d\Omega_{p+2}^2 = \frac{1}{1-\rho^2}d\rho^2 + (1-\rho^2)d\phi^2 + \rho^2 d\Omega_p^2 \,, \tag{4.2}$$

where, $d\Omega_p^2$, refers to the standard metric of the S^p sphere, which we take to be parametrised by the angles $\theta_1, \theta_2, \ldots, \theta_{p-1}, \psi$ with $0 \leq \theta_i \leq \pi$ and $0 \leq \psi \leq 2\pi$.

Following [6] we now consider configurations in which the p-brane wraps the S^p sphere. We choose a static gauge where the time parameter of the worldvolume $\tau = t$ while the p angular spacelike parameters σ_i are set to be equal to the angles on the S^p, $\sigma_i = \theta_i$. The dynamical variables are then $r(t,\theta_i), X^i(t,\theta_i), \rho(t,\theta_i)$ and $\phi(t,\theta_i)$. We consider configurations where these quantities do not depend on θ_i so that there are no brane oscillations. Furthermore since there is complete translational symmetry along X^i the corresponding momenta are conserved. We will study motions where these momenta are identically zero. Motions with non-zero momenta along the brane can be easily obtained by performing boosts. In our ansatz then, the dynamical variables are $r(t), \rho(t)$ and $\phi(t)$. The DBI action is

$$S_{\rm DBI} = -T_p V_p \int dt e^{-\phi}(f(r)\rho^2 r^2)^{p/2}\sqrt{g_{tt} - g_{rr}\dot{r}^2 - g_{\rho\rho}\dot{\rho}^2 - g_{\phi\phi}\dot{\phi}^2} \,, \tag{4.3}$$

where V_p stands for the volume of the p-sphere and we have carried out the integrals along the S^p world volume directions.

In addition the brane action gets a contribution from the Cherns Simon term. This arises because the $(6-p)$-brane gives rise to a magnetic $p+2$ form field strength (or equivalently an electric $F^{(8-p)}_{013,\ldots,(6-p)r}$ field strength) that threads the $p+2$ sphere in (4.1). It is:

$$T_p F_{\rho\phi\theta_1,\ldots,\theta_{p-1}\psi} = \frac{2\pi N}{V_{p+2}}\rho^p \epsilon_{\theta_1,\ldots,\theta_{p-1}\psi} \,, \tag{4.4}$$

[7]The considerations below are valid for all $p > 0$. The case of $p = 0$ is discussed separately at the end of the subsection

where $\epsilon_{\theta_1,\ldots,\theta_{p-1}\psi}$ is the volume form of the unit p-sphere and V_{p+2} denotes the total volume of the unit $p+2$ sphere, respectively. N in (4.4) refers to the number of $(6-p)$-branes. From (4.4) we see that with an appropriate choice of gauge we can take

$$T_p C_{\phi\theta_1,\ldots,\theta_{p-1}\psi} = \frac{2\pi N}{V_{p+2}(p+1)} \rho^{(p+1)} \epsilon_{\theta_1,\ldots,\theta_{p-1}\psi}\,, \tag{4.5}$$

where $\epsilon_{\theta_1,\ldots,\theta_{p-i}}$ is the volume form on a unit S^p sphere.

The Cherns Simon term (after integrating over the S^p world volume directions again) is given by

$$S_{\mathrm{CS}} = \frac{2\pi V_p N}{V_{p+2}(p+1)} \int dt \rho^{p+1} \dot{\phi}\,. \tag{4.6}$$

Now using the fact that

$$V_p = \frac{2\pi^{(p+1)/2}}{\Gamma((p+1)/2)}\,, \tag{4.7}$$

we get the full action from (4.3) and (4.6) to be

$$S = -T_p V_p \int dt e^{-\phi} (f(r)\rho^2 r^2)^{p/2} \sqrt{g_{tt} - g_{rr}\dot{r}^2 - g_{\rho\rho}\dot{\rho}^2 - g_{\phi\phi}\dot{\phi}^2} + N \int dt \rho^{p+1}\dot{\phi}\,. \tag{4.8}$$

To study the resulting dynamics it is useful to construct the hamiltonian for this system.

The momenta are:

$$\begin{aligned}
P_r &= \frac{\partial L}{\partial \dot{r}} = \frac{T_p V_p e^{-\phi}}{\sqrt{g_{tt} - g_{rr}\dot{r}^2 - g_{\rho\rho}\dot{\rho}^2 - g_{\phi\phi}\dot{\phi}^2}} (f\rho^2 r^2)^{p/2} g_{rr}\dot{r}\,,\\
P_\rho &= \frac{\partial L}{\partial \dot{\rho}} = \frac{T_p V_p e^{-\phi}}{\sqrt{g_{tt} - g_{rr}\dot{r}^2 - g_{\rho\rho}\dot{\rho}^2 - g_{\phi\phi}\dot{\phi}^2}} (f\rho^2 r^2)^{p/2} g_{\rho\rho}\dot{\rho}\,,\\
P_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \frac{T_p V_p e^{-\phi}}{\sqrt{g_{tt} - g_{rr}\dot{r}^2 - g_{\rho\rho}\dot{\rho}^2 - g_{\phi\phi}\dot{\phi}^2}} (f\rho^2 r^2)^{p/2} g_{\phi\phi}\dot{\phi} + N\rho^{p+1}\,.
\end{aligned} \tag{4.9}$$

The hamiltonian then is

$$\begin{aligned}
H &= P_r\dot{r} + P_\rho\dot{\rho} + P_\phi\dot{\phi} - L\\
&= \sqrt{g_{tt}} \left[(T_p e^{-\phi} V_p)^2 (f(r)\rho^2 r^2)^p + \frac{P_r^2}{g_{rr}} + \frac{P_\rho^2}{g_{\rho\rho}} + \frac{(P_\phi - N\rho^{p+1})^2}{g_{\phi\phi}} \right]^{1/2}.
\end{aligned} \tag{4.10}$$

Now notice that if

$$T_p e^{-\phi} V_p (f(r) r^2)^{(p+1)/2} = N\,, \tag{4.11}$$

the first and last terms within the square brackets above can be combined, exactly as in [10] and the hamiltonian can be rewritten as

$$H = \sqrt{g_{tt}} \left[\frac{P_\phi^2}{f(r) r^2} + \frac{P_r^2}{g_{rr}} + \frac{P_\rho^2}{g_{\rho\rho}} + \frac{(\rho P_\phi - N\rho^p)^2}{g_{\phi\phi}} \right]^{1/2}, \tag{4.12}$$

where we have used from (4.1) and (4.2) that $g_{\phi\phi} = f(r)r^2(1-\rho^2)$. Equation (4.11) is an important condition and we will refer to it repeatedly in the discussion below.

P_ϕ is a constant of motion. It is clear from (4.12) that for a given P_ϕ the lowest energy configuration satisfies $P_\rho = 0$ for all time. This is because ρ does not appear in the first two terms and the equation of motion for such configurations simply require that the last term vanishes. This gives the equilibrium value of ρ in terms of P_ϕ

$$P_\phi = N\rho^{p-1}\,, \tag{4.13}$$

a condition independent of r. The hamiltonian then reduces to

$$H = \sqrt{g_{tt}}\left[\frac{P_\phi^2}{f(r)r^2} + \frac{P_r^2}{g_{rr}}\right]^{1/2}. \tag{4.14}$$

Now, we come to the punch line of this section. Notice, that (4.14) is exactly the hamiltonian for a massless particle which carries angular momentum P_ϕ on the S^{p+2} sphere — one simple way to see this is to consider the laplacian in the WKB approximation. Thus, as long as (4.11) is met, the expanded p-brane behaves like a massless particle.

However, unlike a usual massless particle the brane has a bounded angular momentum for such motions, just as in [6]. This follows from (4.13). Since $0 < \rho < 1$ the maximum angular momentum is N. This is the analog of the stringy exclusion principle.

It important to note that the physical size of the brane in the string metric

$$R = f^{1/2} r\rho \tag{4.15}$$

depends on r and hence is not a constant of motion. However, this is entirely due to the change in radius of the transverse $(p+2)$ sphere.

We now examine specific examples to see when (4.11) is met. First, consider the near horizon geometry for the extremal D$(6-p)$-brane [20]. In this case

$$\begin{aligned}
g_{tt} &= g_{ii} = H^{-1/2}\,,\\
g_{rr} &= H^{1/2}\,,\\
f(r) &= H^{1/2}\,,\\
e^{\phi} &= H^{(p-3)/4}\,,\\
\text{where}\qquad H &= \left(\frac{R}{r}\right)^{p+1},\\
\text{and}\qquad R^{p+1} &= 2^{p-1}\pi^{(p-1)/2}\,\Gamma\left(\frac{p+1}{2}\right) g_s l_s^{p+1} N\,,
\end{aligned} \tag{4.16}$$

where N is the number of D$(6-p)$-branes. We see then that (4.11) is indeed met. We would like to emphasize that for $p \neq 3$ the near extremal geometry is not AdS. Next consider the non-extremal, near horizon geometry [21]. In this case g_{tt}, g_{rr} are different from their values in the extremal case, but $f(r), e^{\phi}$, and H are still unchanged from (4.16) so that once again (4.11) is met. The hamiltonian is again given by (4.12) with the appropriate g_{rr} and g_{tt}. This illustrates that even in non-supersymmetry preserving backgrounds the expanded brane can behave like a massless particle.

Our conclusions above also apply to non extremal, near horizon M2- and M5-brane metrics. In fact, the discussion above can be carried over to those cases almost directly. Let us briefly sketch out how this happens. The total number of dimensions now is eleven and in (4.1) the sum over coordinates parallel to the brane goes from $i = 1$ to $(7-p)$. The metric has an SO$(p+2)$ symmetry. The case $p = 5$ with SO(7) symmetry refers to the M2-brane background while case $p = 2$ with SO(4) symmetry to the M5-brane background. The dynamics of the p-brane moving in this background is described by an action consisting of a BI term and a Cherns Simon term. The BI term is given by (4.3) without the dilaton factor $e^{-\phi}$, where T_p stands for the tension of the p-brane, while the CS term is still (4.6), with N being the number of $(7-p)$-branes. The crucial condition (4.11) is now replaced by

$$T_p V_p (R_{\mathrm{AdS}})^{p+1} = N\,. \tag{4.17}$$

One can see that this is met for the M2- and M5-brane geometries. Moreover, it is independent of whether we are dealing with the extremal or non-extremal cases. Thus for both these cases the special solutions which satisfy (4.13) (with $p = 5$ and $p = 2$, respectively) yield an energy,

$$H = \sqrt{g_{tt}} \left[\frac{P_\phi^2}{R_{\mathrm{AdS}}^2} + \left(\frac{r}{R_{\mathrm{AdS}}} \right)^{(p+1)/2} P_r^2 \right]^{1/2}, \tag{4.18}$$

which is the same as that for a massless particle.

Two special cases $p = 1$ and $p = 0$ in the discussion above are worth commenting on. In the $p = 1$ case, (4.13) is independent of ρ and the special solution, with an energy equal to the massless case, exists only for

$$P_\phi = N\,. \tag{4.19}$$

Moreover, if (4.19) is true the potential for ρ vanishes. Thus, for this special value of angular momentum there is a one-parameter family of solutions all of which behave like massless particle. These comments are equally valid in the extremal and non-extremal cases. A closely related case is realised when one considers a fundamental string in an NS 5-brane background. In this case the near-horizon geometry has a three-sphere of constant radius R_3. (4.11) is replaced by a similar condition which does not involve the dilaton and relates R_3 to the number of five-branes. The condition is in fact met leading again to special solutions for (4.19) which behave like massless particles both in the extremal and non-extremal cases.

For $p = 0$ the formulae above receive some modifications. First, the last term in the metric on S^{p+2} in (4.2) is not present. The coordinate ρ can still be used.[8] Secondly in (4.6) one has to define $V_0 = 1$. As a result the coefficient of the Chern-Simons term in (4.8) acquires an additional factor of 1/2. This modifies the condition (4.11) to

$$T_0 e^{-\phi}(f(r)r^2)^{1/2} = \frac{1}{2}N\,, \qquad (p=0)\,. \tag{4.20}$$

One can verify (4.16) that this is indeed met for the six-brane extremal and non-extremal, near-horizon metrics. This case, differs physically from $p > 0$ in some important ways. Note that for $p \geq 1$ the brane in question has *no net charge* since it is always wrapped on a S^p which is contained in the S^{p+2}. However a single zero brane carries charge and consequently does not have the same quantum numbers as a graviton. One possiblility is to consider a pair of a zero brane and an anti zero brane — this would have a state with the same energy as a pair of gravitons. Furthermore the relationship which determines the equilibrium value of ρ is pathological since it implies that there is a *lower* bound on the angular momentum. Some of these problems can be possibly resolved by introducing a coupling between the D0-$\bar{\text{D}}$0 pair [6].

Finally, consider for example the extremal Dp-brane geometry without taking the near horizon limit. In this case the Harmonic function in (4.16) is replaced by $H = 1 + (R/r)^{p+1}$. Now one can verify that (4.11) is no longer met. For example in the asymptotic region the dilaton and $f(r)$ go to constant so the l.h.s. of (4.11) grows like r^{p+1}. In fact one show in this case that the energy of the expanded brane configuration is always bigger than for a massless particle. Furthermore it is no longer possible to obtain motions with $P_\rho = 0$ with non-zero ρ, a fact which may be easily seen in the deep asymptotic region of large r.

4.2 4D and 5D black holes

The condition (4.11) (or (4.20) for $p = 0$) is also satisfied by appropriate branes moving in the near horizon geometries of five- and four-dimensional black holes in string theory. As in the previous subsection we have to discuss the motion of branes which are *magnetic duals* of the branes which produce the background.

Consider first the 5D extremal black hole obtained in IIB string theory compactified on a $\mathbb{T}^4 \times S^1$ with Q_5 D5-branes wrapping $\mathbb{T}^4 \times S^1$ and Q_1 D1-branes wrapping the S^1. The magnetically dual branes are then, (i) a D1-brane wrapping a circle on the transverse S^3 which can couple to the mangetic 3-form field strength threading the S^3 and (ii) a D5-brane wrapped on $\mathbb{T}^4$ and a circle on the S^3. These are further wrapped on a circle on the transverse S^3 and move on it. Both cases relate to the $p = 1$ case of the previous subsection with N replaced by Q_5 for (i) and N replaced

[8] Setting $\rho = \cos\theta$ yields the familiar metric on the two-sphere.

by Q_1 for case (ii). Using the well-known background geometry [22] it is straightforward to check that (4.11) is satisfied. The only values of the angular momentum for which one has equilibrium brane configurations with the same energy as gravitons are Q_5 and Q_1, respectively. In this case the geometry is in fact $AdS_3 \times S^3 \times \mathbb{T}^4$ (which has been considered in [6]) and the exclusion principle bound is $Q_1 Q_5$ which differs from both these values.[9]

Similarly the four-dimensional black hole in IIA string theory compactified on $\mathbb{T}^4 \times S^1 \times \tilde{S}^1$ is made of Q_2 D2-branes wrapping $S^1 \times \tilde{S}^1$, Q_6 D6-branes wrapping $\mathbb{T}^4 \times S^1 \times \tilde{S}^1$ and Q_5 NS5-branes wrapping $\mathbb{T}^4 \times \tilde{S}^1$. For extremal black holes the geometry is $AdS_2 \times S^2 \times \mathbb{T}^6$. Now the magnetically dual objects are ; (i) D4-branes wrapping the $\mathbb{T}^4$ (ii) D0-branes, and (iii) $F1$ string wrapping S^1. They all move on the transverse S^2. All these relate to the $p = 0$ case discussed in the previous section with the coefficient of the Chern Simons term replaced by $\frac{1}{2}Q_2$ for (i), by $\frac{1}{2}Q_6$ for (ii) and $\frac{1}{2}Q_5$ for (iii). Once again it may be verified that (4.20) is satisfied for all the cases (for case (iii) the dilaton factor is absent in (4.20), as commented above).

In both these cases the addition of a momentum along the S^1, or addition of non extremality does not change the result since they do not affect the S^3 or S^2 parts of the metric, respectively.

4.3 Discussion

It is worth discussing the results of the above calculation in some detail.

Let us begin by relating the discussion of the previous section to [6]. The discussion in [6] was for AdS space and overlaps with the analysis above in the the D3-brane, M2- and M5-brane cases. The one difference is that we have used Poincare coordinates instead of Global ones. The hamiltonian for the M2- and M5-branes is given in (4.18). For the D3-brane case we have from (4.14) and $p = 3$

$$H = \frac{r}{R}\left[\frac{P_\phi^2}{R^2} + \frac{R^2}{r^2}P_r^2\right]^{1/2} . \tag{4.21}$$

The prefactor $\sqrt{g_{tt}}$ in (4.18) and (4.21) is the usual red-shift in energy. Due to it we see that a massless particle, or equivalently an expanded brane, initially at rest in the radial direction will fall into the black hole. In contrast, in global coordinates a particle at the center of AdS does not move and the energy in global coordinates is equal (in units of the radius) to the angular momentum, making the BPS nature of the state more transparent.

For extremal D-brane backgrounds, the equation of motion which follows from the hamiltonian (4.14) can be written as

$$(\dot{r})^2 + U(r, P_\phi, E) = 0\,, \tag{4.22}$$

[9] We would like to thank S.D. Mathur for discussions about this point

where E is the energy and

$$U(r, P_\phi, E) = \frac{P_\phi^2}{E^2}\, r^{2p} - r^{p+1}\,. \tag{4.23}$$

It is thus clear that motion is always restricted between $r = 0$ and a turning point

$$r_t = \left(\frac{E}{P_\phi}\right)^{2/(p-1)}\,. \tag{4.24}$$

So far as motion in the radial direction is concerned the angular momentum provides a potential well which prevents the particle to escape to large r. For non-extremal near-horizon D-brane backgrounds, however, the angular momentum provides a finite potential *barrier* near the horizon, just as in the vicinity of Schwarzschild black holes.

In the non-AdS extremal backgrounds discussed above, the expanded p-brane solution is not supersymmetric, as best as we can tell. Certainly this is true in the non-extremal geometries. Despite this one finds that the expanded branes behave like massless particles as long as (4.11) or (4.17) is met. Unfortunately, we do not understand the significance of this condition well enough at the moment. One comment is worth making in this context though. Consider the Dp-brane case first. Due to the factor $e^{-\phi}$ multiplying the BI action, the metric seen by the Dp-brane differs from the string metric by a conformal factor. It is

$$ds_p^2 = (e^{-2\phi})^{1/(p+1)}\, ds^2_{\rm string}\,. \tag{4.25}$$

Interestingly, the resulting metric is AdS space. Eq. (4.11) then states that the radius of the $p+2$ sphere, in the p-brane metric, must be a constant and determined by the magnetic flux. For the $M2$, $M5$ brane cases no rescaling is required and the p-brane metric is the M theory metric. (4.17) then sets the radius of the $p+2$ sphere equal to an appropriate constant.

We noted before that the equilibrium size of the p-brane *in the string metric* is not a constant of motion. The above considerations, however, show that the equilibrium size *in the p-brane metric* is indeed a constant of motion.

Physically, it seems puzzling that an extended brane configuration manages to have the same energy as a massless particle. The answer lies in the fact that the expanded p-brane is the magnetic dual of the $(6-p)$ branes which gives rise to the background. The resulting Cherns Simon coupling reduces the energy required, for fixed angular momentum, by just the right amount to equal the extra potential energy needed to support the extended brane.

The special solutions (4.13), (4.14) and (4.18), exist only when the angular momentum is less than N. For higher angular momenta, it is safe to conjecture that there is no expanded brane configuration which behaves like a massless particle. This is the analogue of the stringy exclusion principle.

We now turn to examining two more issues in some detail. In the next section we show that the graviton and the expanded brane descriptions are valid for different and non-overlapping ranges of angular momentum. This is important if expanded branes are to be identified with gravitons. In the last section of the paper we focus on one special instance of the general discussion above: the $p = 2$ case. In this case we have a D2-brane expanded into a two-sphere in the D4-brane background. One can in addition turn on N_0 units of magnetic flux on the world volume of the D2-brane. We show that for an appropriate region of parameter space this configuration can be described as N_0 D0-branes, puffed up into a non-commutative two-sphere, and rotating on the S^4.

5. Gravitons vs expanded branes

In general one would expect that the descriptions in terms of a graviton and an expanded brane state are vaild in different regions of parameter space. Certainly one can argue this for the AdS backgrounds studied in [6] where the BPS nature of the states ensures there cannot be multiple copies. But even more generally for the p-brane extremal and non-extremal backgrounds one expects only one of the two description to be valid.

We will now argue that this is indeed the case. The graviton and expanded brane description are valid for different values of angular momentum.[10]

Let us start with the case of an $AdS_m \times S^{p+2}$ background in M-theory. In analyzing the graviton states one can think of doing a Kaluza Klein reduction on the S^{p+2}. The graviton then turns into a massive state with mass

$$M \sim \frac{P_\phi}{R_{\rm AdS}} \sim M_{\rm Pl} \frac{P_\phi}{N^{1/(p+1)}}\,, \tag{5.1}$$

where P_ϕ refers to the angular momentum, and we have used the fact that

$$R_{\rm AdS} \sim \frac{N^{1/(p+1)}}{M_{\rm Pl}}\,. \tag{5.2}$$

In order to neglect the higher derivative terms in the action, arising for example from higher powers of the curvature, and treat the graviton in a controlled manner we need

$$M \ll M_{\rm Pl}\,, \tag{5.3}$$

leading to

$$P_\phi \ll N^{1/(p+1)}\,. \tag{5.4}$$

The alternative description in this case involves an expanded p-brane. This description in under control when the brane has a big size compared to the Planck Scale so

[10]Here angular momentum refers to rotations on SO$(p + 2)$.

that acceleration terms can be neglected and one can work with the BI + CS action. This gives a condition

$$R_{\mathrm{AdS}}\,\rho \gg \frac{1}{M_{\mathrm{Pl}}}\,. \tag{5.5}$$

Substituting for ρ from (4.13) we get

$$P_\phi \gg N^{2/(p+1)}\,. \tag{5.6}$$

We see that (5.3) and (5.6) can never be simultaneously met.

Before proceeding let us make two comments. First, our use of the word graviton should not be taken literally. We simply mean a fluctuation about the $AdS_m \times S^{p+2}$ supergravity background which is massless in 11 dim. Second, the Planck scale in (5.3) is the 11 dim. Planck scale. The gravitational backreaction after Kaluza Klein reduction is governed by the $(9-p)$-dimensional Planck scale, which is bigger than M_{Pl} since $R_{\mathrm{AdS}}\,M_{\mathrm{Pl}} \gg 1$. Requiring these to be under control, therefore, is a less stringent condition than (5.3).

The D3-brane case is similar to the case above with the string scale playing the role of M_{Pl}. The general Dp-brane case has one new aspect: the radius of the S^{p+2} is not constant in these cases. Inspite of this the argument above essentially goes through. Consider a massless particle moving on the S^{p+2}. Carrying out a Kaluza Klein reduction on the $p+2$ sphere and demanding that the resulting mass is smaller than the string scale yields the condition

$$P_\phi \ll \frac{R_{p+2}}{l_s}\,, \tag{5.7}$$

where R_{p+2} is the radius of the $p+2$ sphere. On the other hand for the expanded Dp-brane description to be valid we have

$$R_{p+2}\,\rho = R_{p+2}\left(\frac{P_\phi}{N}\right)^{1/(p-1)} \gg l_s\,. \tag{5.8}$$

One can show that (5.7) and (5.8) cannot be simulateneously valid if the dilaton $e^\phi \ll 1$ and string loop corrections to the supergravity description are under control.

To see this note that

$$R_{p+2} = \left(\frac{R}{r}\right)^{(p+1)/4} r = \left(\frac{R}{r}\right)^{(p-3)/4} R\,. \tag{5.9}$$

So that the dilaton, (4.16), can be expressed as

$$e^\phi = \left(\frac{R}{r}\right)^{(p+1)(p-3)/4} = \left(\frac{R_{p+2}}{R}\right)^{p+1}\,. \tag{5.10}$$

Using the fact that $R^{p+1} \sim g_s N l_s^{p+1}$ one can also express this as

$$e^\phi \sim \frac{1}{g_s N}\left(\frac{R_{p+2}}{l_s}\right)^{p+1}\,. \tag{5.11}$$

Now, if (5.7) and (5.8) are simulateneously valid,

$$\left(\frac{R_{p+2}}{l_s}\right) \gg N\left(\frac{l_s}{R_{p+2}}\right)^{p-1} . \tag{5.12}$$

But then it follows from (5.11) that

$$e^{\phi} \gg \left(\frac{1}{g_s N}\right) N^{(p+1)/p} = \left(\frac{1}{g_s}\right) N^{1/p} \gg 1\,, \tag{5.13}$$

where the last inequality arises because $g_s \to 0$ and $N \to \infty$. Thus in conclusion, when the supergravity approximation is valid, the graviton and expanded brane description are never simultaneously valid.

Let us comment on condition (5.8) in some more detail. Since R_{p+2} depends on r the massless particle after KK reduction gets a position dependent mass. In other words, in the KK reduced theory the particle satisfies a wave equations with a potential energy term. If this potential energy is of order the $M_{\rm Pl}$ higher derivative terms will be important leading to (5.8).

In summary then, we have seen above that the massless particle description and the expanded brane description are valid for different values of the angular momentum. As the rotational energy for the graviton increases and becomes larger than the string scale (or Planck scale in M-theory) the gravitons turn into an expanded brane configuration. This is made all the more plausible by the fact that in several cases even without supersymmetry the expanded brane solutions has the same energy, for fixed angular momentum, as the massless particle. Once we accept this identification it can be extended to other cases, where the expanded brane has a different energy from the massless particle. For example, one can consider the expanded brane moving in the full $(6-p)$-brane geometry. Close to the horizon it behaves like a massless particle, but the identification should still be valid as it moves further away.

6. Puffed rotating D0-branes

In this section we return to considering one special case of the general discussion in section 4: a D2-brane moving in the background of the D4-brane. This corresponds to $p = 2$; the background geometry has a SO(4) rotational symmetry in this case. We showed in section 4 that when the D2-brane carries SO(4) angular momentum there is a particular solution (4.13), for which it behaves, in effect, like a massless particle (4.14). The D2-brane in this configuration expands into a two-sphere. Here we consider what happens when in addition N_0 units of magnetic flux are turned on in the world volume of the D2-brane. Through the usual Cherns Simons coupling it then acquires N_0 units of D0-brane charge. We will see below that there is another solution consisting of N_0, D0-branes, also carrying the same angular momentum, in which the D0-branes have puffed up into a non-commutative two-sphere. Thus, we have another example of the Magnetic Moment effect discussed in section 2.2, but this time in a non-constant four-form field generated by a D4-brane background.

6.1 D2-brane with U(1) flux

To keep the discussion simple, we focus on the near-horizon extremal D4-brane background. This is given by the metric and dilaton:

$$
\begin{aligned}
ds^2 &= H^{-1/2}\left(-dt^2 + \sum_{i=1}^{4}(dX^i)^2\right) + H^{1/2}\sum_{i=5}^{9}(dX^i)^2\,, \\
e^{\phi} &= H^{-1/4}\,.
\end{aligned}
\tag{6.1}
$$

Here $X^i, i = 5,\ldots,9$ denote the five transverse coordinates, $r^2 = \sum_{i=5}^{9}(X^i)^2$ and $H = (R/r)^3$. To relate this to the metric (4.1) and (4.2) we need the following relations:

$$
\begin{aligned}
X^5 &= r\sqrt{1-\rho^2}\cos\phi\,, \qquad X^6 = r\sqrt{1-\rho^2}\sin\phi\,, \\
X^7 &= r\rho\cos\theta\,, \qquad X^8 = r\rho\sin\theta\sin\psi\,, \\
X^9 &= r\rho\sin\theta\cos\psi\,.
\end{aligned}
\tag{6.2}
$$

This gives rise to the metric

$$
\begin{aligned}
ds^2 = H^{-1/2}(-dt^2 + (dX^i)^2) + H^{1/2}\bigg[dr^2 &+ \frac{r^2}{1-\rho^2}d\rho^2 + r^2(1-\rho^2)d\phi^2 + r^2\rho^2 d\theta^2 \\
&+ r^2\rho^2\sin^2\theta d\psi^2\bigg]\,,
\end{aligned}
\tag{6.3}
$$

which agrees with (4.1) and (4.2).

With N_0 units of magnetic flux the DBI action for the D2-brane, (4.3), is replaced by

$$
\begin{aligned}
S_{\mathrm{DBI}} = -T_2 4\pi \int dt e^{-\phi}\left[(H^{1/2}\rho^2 r^2)^2 + \frac{N_0^2\lambda^2}{4}\right]^{1/2} &\times \\
\times \sqrt{g_{tt} - g_{rr}(\dot{r})^2 - g_{\rho\rho}(\dot{\rho})^2 - g_{\phi\phi}\dot{\phi}^2}&\,.
\end{aligned}
\tag{6.4}
$$

The Cherns Simon term involving the coupling to the four form is left unaltered, (4.6). For studying the dynamics, most of the discussion, (4.3)–(4.9), can be carried over with only slight modifications. Once again focusing on the special case when $P_\rho = 0$ and (4.13) is met yields a hamiltonian:

$$
H = \sqrt{g_{tt}}\left[\left\{4\pi T_2 e^{-\phi}\left(\frac{N_0\lambda}{2}\right)\right\}^2 + \frac{P_\phi^2}{H^{1/2}r^2} + \frac{P_r^2}{g_{rr}}\right]^{1/2}\,.
\tag{6.5}
$$

JHEP10(2000)037

The reader will notice that this is the hamiltonian for a particle moving in the four-brane background with a with a position dependent mass

$$m = 2\pi T_2 \lambda N_0 e^{-\phi}\,. \tag{6.6}$$

Using the relation $2\pi T_2\lambda = T_0$, we see that this mass is identical to that of N_0 D0-branes. So the expanded D2-brane solution with N_0 units of magnetic flux has an energy exactly equal to N_0 D0-branes executing only center of mass motion, with no relative displacement. However, our experience in other situations discussed above would make us suspect that there is another solution for N_0 D0-branes involving a non-commutative two-sphere and this is the solution to be identified with the expanded D2-brane.[11]

6.2 Puffed rotating D0-branes.

This expectation is indeed correct. To verify it we need to consider the non-abelian D0-brane lagrangian in the curved D4-brane geometry. The abelian BI lagrangian, in static gauge, is given by:

$$L = -T_0 N_0 e^{-\phi}\sqrt{g_{tt} - g_{ij}\dot{x}^i\dot{x}^j}\,. \tag{6.7}$$

This suggests that the non-abelian lagrangian (upto quartic terms) is given[12] by

$$\begin{aligned} L = -T_0\,\mathrm{Tr}\Bigg[e^{-\phi(X)}\sqrt{g_{tt}}\Bigg\{1 - \frac{1}{2}\frac{g_{ij}(X)}{g_{tt}(X)}\dot{X}^i\dot{X}^j - \frac{1}{4\lambda^2}\sum_{ab}[X^a,X^b][X^c,X^d]\ \times \\ \times\, g_{ac}(X)g_{bd}(X)\Bigg\}\Bigg] + i\frac{T_0}{\lambda}\,\mathrm{Tr}[C^3_{ijk}(X)X^iX^j\dot{X}^k]\,. \end{aligned} \tag{6.8}$$

Notice since the background is space-time dependent the background fields lie within the matrix traces above. The last term in (6.8) is a Cherns Simon coupling which arises as discussed in [8] and involves the RR 3 form gauge potential C^3. One big difference between the discussion here and in section 2.2 is that the four form field strength is not constant. Consequently we need to work with the full gauge potential C^3 in (6.8) rather than its expansion to linear order.

Motivated by the D2-brane solution discussed above and in section 4 we consider the following ansatz for D0-brane solution

$$\begin{aligned} X^5 &= r\sqrt{1-\rho^2}\cos\phi\,\mathbb{1}\,, \qquad X^6 = r\sqrt{1-\rho^2}\sin\phi\,\mathbb{1}\,, \\ X^{i+6} &= \frac{2}{N_0}r\rho\ J^i\,, \qquad \{i=1,2,3\}\,, \end{aligned} \tag{6.9}$$

[11]For example the expanded D2-brane configuration has dipole moment with respect to F^4 as does the puffed up D0-brane configuration but not the D0-brane configuration with no relative displacement.

[12]We note that this lagrangian was also considered in [18].

where J^i stand for SU(2) generators in the N_0-dimensional irreducible representation. Further, we take r, ϕ to be time dependent and take ρ to be time independent. All the other coordinates, parallel to the 4-brane are taken to be a constant multiple of the identity. It is worth pointing out that the coordinates X^7, X^8, X^9 do not commute and form a non-commutative two sphere; further,

$$\begin{aligned} (X^7)^2 + (X^8)^2 + (X^9)^2 &= r^2\rho^2 \mathbb{1}\,, \\ (X^5)^2 + (X^6)^2 + (X^7)^2 + (X^8)^2 + (X^9)^2 &= r^2 \mathbb{1}\,. \end{aligned} \tag{6.10}$$

Now notice that the metric coefficients and the dilaton dependence in (6.1) are a function of r alone. In the lagrangian (6.8) r^2 is to be replaced by

$$r^2 \longrightarrow \sum_{i=5}^{9} (X^i)^2\,. \tag{6.11}$$

Luckily, due to (6.10) this is a multiple of the identity matrix and can be taken out of the trace and replaced by the c-number r^2. Thus we can take all the dependence on the background metric and dilaton outside the matrix traces in (6.8). This leads to considerable simplification in evaluating the lagrangian.

To evaluate the CS term we need the three form potential C^3 in the coordinates (6.1). C^3 in the coordinates (4.1) and (4.2) was determined in (4.5). Using (6.2) to change coordinates we get,

$$\begin{aligned} T_0 C_{578} &= -\frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \sin\phi X^9\,, \qquad & T_0 C_{589} &= -\frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \sin\phi X^7\,, \\ T_0 C_{597} &= -\frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \sin\phi X^8\,, \qquad & T_0 C_{678} &= \frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \cos\phi X^9\,, \\ T_0 C_{689} &= \frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \cos\phi X^7\,, \\ T_0 C_{697} &= \frac{N}{2}\frac{1}{r^3}\frac{1}{\sqrt{r^2 - r^2\rho^2}} \cos\phi X^8\,, \end{aligned} \tag{6.12}$$

all other components are zero. We also remind the reader that N in (6.12) refers to the number of D4-branes whereas N_0 stands for the number of D0-branes. Once again, in the Chern Simon term, strictly speaking all the space dependence in C^3 should be replaced by functions of the coordinate matrices X^i. However due to (6.10) and the argument given above the r and $r\rho$ dependence can continue to be regarded as c numbers. Similarly, since ϕ can be expressed in terms of X^5, X^6 alone and both of these are multiples of the identity it too can be regarded as a c number. This greatly simplifies the evaluation of the CS term. Each component of C^3 now gives a term proportional to $\mathrm{Tr}(X^7[X^8, X^9])$ which is proportional to the simplectic two-form on the two-sphere.

JHEP10(2000)037

Putting all this together finally yields a lagrangian:

$$L = -N_0 T_0 e^{-\phi}\sqrt{g_{tt}}\left[1 - \frac{1}{2}\frac{g_{rr}}{g_{tt}}\dot{r}^2 - \frac{1}{2}\frac{g_{\phi\phi}}{g_{tt}}\dot{\phi}^2 + \frac{2}{N_0^2\lambda^2}Hr^4\rho^4\right] + N\dot{\phi}\rho^3 \,. \tag{6.13}$$

To compare this with the D2-brane action we expand (6.4) in the non-relativistic limit and assume that the N_0 units of magnetic field dominates the action compared to the surface tension term. One gets on keeping the leading term and the first correction (and after the identification $2\pi\lambda T_2 = T_0$) exactly (6.13). Since the the two lagrangians agree, one can use our discussion in section 4 for the D2-brane case to conclude again that minimizing with respect to ρ yields the condition (4.13). Substituting this in the resulting hamiltonian yields:

$$H = \sqrt{g_{tt}}\left[N_0 T_0 e^{-\phi} + \frac{1}{2}\frac{1}{N_0 T_0 e^{-\phi}}\left(\frac{P_\phi^2}{H^{1/2}r^2} + \frac{P_r^2}{g_{rr}}\right)\right] . \tag{6.14}$$

This is the non-relativistic version of (6.5) and corresponds to a non-relativistic particle of mass $NT_0 e^{-\phi}$ moving in the D4-brane background.

In summary, we have found a solution to the non-abelian D0-brane action in which the D0-branes rotating in the presence of the D4-brane background, puffs up into a non-commutative two-sphere. The solution carries exactly the same energy as if only the center of mass of the D0-branes was moving with no relative displacement. There is also a expanded D2-brane solution with the same quantum numbers and the same energy.

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